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One-loop effective potentials in quantum electrodynamics

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Abstract. We investigate the one-loop effective potential for some typical external fields in electrodynamics, in particular for a constant magnetic field and a laser field. Our treatment is based on the corresponding Green function as explicit functional of the prescribed field. Schwinger's source- and proper-time techniques will be used throughout.

1. Introduction

The aim of this paper is to illustrate the computation of one-loop effective potentials in quantum electrodynamics (QED). Different types of external electromagnetic fields will be studied. We demonstrate that the existence of non-linear vacuum phenomena, pair production, etc depends essentially on the nature of the external prescribed field.

The vacuum persistence amplitude $\langle 0_+|0_-\rangle^A = \exp(\mathrm{i}W^{(1)}[A])$ will be exclusively the quantity of interest. It summarizes the effect that an arbitrary number of external photon lines can have on a single fermion loop. Since the action $\mathrm{i}W^{(1)}[A]$ is directly related to the Green function $G_+[A]$, we are faced primarily with the question of how to find $G_+[A]$. Being interested in vacuum polarization phenomena only, we want to compute $G_+(x, y|A)$ for $x \simeq y$, i.e., in the neighbourhood about where the quantum mechanical fluctuations take place. This summarizes all the information necessary to compute the one-loop correction to the classical Lagrangian. It will prove useful to work side-by-side with various representations for $G_+[A]$ and $\mathrm{i}W^{(1)}[A]$.

Besides the space and momentum representation, it is most convenient to exhibit those quantities in the proper-time formalism which makes it easy to compute the necessary traces.

In § 2, we shall collect the functional ingredients and then proceed in § 3 with vacuum polarization effects for different types of electromagnetic fields.

2. Functional statements

Here we want to collect the relevant Green function equation, closed-loop factor, etc for spinor QED (Schwinger 1973, Fried 1972). The process which summarizes the effect that an external field environment $A(x)$ can have on a single fermion loop, is given analytically by

$$\langle 0_+|0_-\rangle^A = \exp(\mathrm{i}W^{(1)}[A]), \tag{2.1}$$

where

$$\mathrm{i}W^{(1)}[A] = -\mathrm{Tr}\ln(1-e\gamma.AG_+)^{-1}, \tag{2.2}$$

$$= -\mathrm{Tr}\ln(G_+[A]/G_+[0])^{-1}, \tag{2.3}$$

and

$$G_+[A] = G_+(1-e\gamma.AG_+)^{-1}. \tag{2.4}$$

Tr indicates the diagonal summation in coordinate and spinor space. Furthermore we have the fundamental Green function equation

$$[m+\gamma(\mathrm{i}^{-1}\partial_x - eA(x))]G_+(x,x'|A) = \delta(x-x'). \tag{2.5}$$

Equation (2.3) tells us that knowledge of $G_+[A]$ suffices to compute $\mathrm{i}W^{(1)}[A]$. For further purposes it is useful to state the functional derivative of $\mathrm{i}W^{(1)}[A]$,

$$\mathrm{i}(\delta W^{(1)}[A]/\delta A_\mu(x)) = -e\ \mathrm{tr}[\gamma_\mu G_+(x,x|A)], \tag{2.6}$$

which follows from (2.2) when cast into the form

$$\mathrm{i}W^{(1)}[A] = -\int_0^e \mathrm{d}e'\ \mathrm{tr}\left(\int (\mathrm{d}x)\gamma_\mu A^\mu(x)G_+(x,x|e'A)\right). \tag{2.7}$$

There is still another useful expression of $W[A]$. If we employ the proper-time representation (Schwinger 1951) for $G_+[A]$:

$$\begin{aligned} &G_+[A] = (m-\gamma\pi)\mathrm{i}\int_0^\infty \mathrm{d}s\ \exp\{-\mathrm{i}s[m^2-(\gamma\pi)^2]\} \\ &\pi_\mu = \mathrm{i}^{-1}\partial_\mu - eA_\mu, \qquad -(\gamma\pi)^2 = \pi_\mu^2 - \tfrac{1}{2}e\sigma_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{2.8}$$

and use the differential version (2.6), we find

$$\mathrm{i}\int \mathscr{L}^{(1)}(\mathrm{d}x) = \mathrm{i}W^{(1)} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\exp(-\mathrm{i}sm^2)\ \mathrm{Tr}\{\exp[\mathrm{i}s(\gamma\pi)^2]\}. \tag{2.9}$$

With the aid of equation (2.6) and (2.9) it is an easy matter to verify that

$$\mathrm{i}(\partial\mathscr{L}^{(1)}/\partial m) = \mathrm{tr}\ G_+(x,x|A). \tag{2.10}$$

Formula (2.10) allows us to obtain the one-loop contribution by a simple integration. All we need is an explicit expression for $G_+(x,x|A)$. This will be the main goal of the next section.

3. Effective Lagrangians

Our starting point is the Green function equation

$$(\gamma\pi+m)G_+(x,x'|A) = \delta(x-x'). \tag{3.1}$$

It is a well known fact that this equation allows only for a closed-form solution for some special cases of the external electromagnetic field $A_\mu(x)$. In what follows we want to concentrate on a constant magnetic (electric) field and a laser field within both scalar and spinor QED. Schwinger solved these problems more than two decades ago with proper-time techniques (Schwinger 1951, Heisenberg and Euler 1936, Weisskopf

1936). Here we give another way to derive the effective Lagrangian for a constant field (Brown and Duff 1975)†.

The basic equation to be solved is

$$[-\partial^2+m^2-\tfrac{1}{2}e\sigma F-\tfrac{1}{4}e^2(x-x')^\mu F^2_{\mu\nu}(x-x')^\nu]\Delta_+(x,x'|A')=\delta(x-x') \tag{3.2}$$

$$A'_\mu=-\tfrac{1}{2}(x-x')^\nu F_{\mu\nu}.$$

The two different Green functions occurring in (3.1) and (3.2) are related by

$$G_+(x,x'|A)=\exp\left(ie\int_{x'}^{x}d\xi^\mu A_\mu(\xi)\right)[m-\gamma^\mu(i^{-1}\partial^x_\mu+\tfrac{1}{2}e(x-x')^\nu F_{\mu\nu}]\Delta_+(x,x';A'). \tag{3.3}$$

Here we represented $G_+[A]$ in such a way that the gauge sensitivity becomes explicit.

Introducing the momentum description by

$$\Delta_+(x,x|A')=\int\frac{dp}{(2\pi)^4}\exp[ip(x-x')]\Delta_+(p|A')$$

we obtain the Green function equation in momentum space:

$$\left(p^2+m^2-\frac{e}{2}\sigma\,.\,F+\frac{e^2}{4}\frac{\partial}{\partial p_\mu}F^2_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)\Delta'_+(p)=1. \tag{3.4}$$

Writing $\kappa^2=m^2-\frac{1}{2}e\sigma F$ and $e^2F^2_{\mu\nu}\equiv e^2F_{\mu\lambda}F^\lambda{}_\nu=f^2_{\mu\nu}$, we get

$$\left(p^2+\kappa^2+\frac{1}{4}\frac{\partial}{\partial p_\mu}f^2_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)\Delta'_+(p)=1. \tag{3.5}$$

This equation can be solved with the *ansatz*

$$\Delta'_+(p;\kappa^2)=i\int_0^\infty ds\,e^{-i\kappa^2 s}\,e^{-M(is)} \tag{3.6}$$

$$=\int_0^\infty ds\,e^{-\kappa^2 s}e^{-M(s)}, \tag{3.7}$$

where

$$M(s)=p^\rho X_{\rho\sigma}(s)p^\sigma+Y(s) \tag{3.8}$$

upon changing $s\to -is$. So we need to evaluate

$$\left(p^2+\kappa^2+\frac{1}{4}\frac{\partial}{\partial p_\mu}f^2_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)\int_0^\infty ds\,e^{-s\kappa^2}\,e^{-M(s)}=1. \tag{3.9}$$

Since

$$\frac{\partial M}{\partial p_\nu}=2p^\rho X_{\rho\nu},\qquad \frac{\partial}{\partial p_\mu}\frac{\partial M}{\partial p_\nu}=2X_{\mu\nu} \tag{3.10}$$

we find

$$\frac{1}{4}\frac{\partial}{\partial p_\mu}f^2_{\mu\nu}\frac{\partial}{\partial p_\nu}e^{-M}=[-\tfrac{1}{2}\mathrm{Tr}(f^2X)+p\,.\,(Xf^2X)\,.\,p]\,e^{-M}. \tag{3.11}$$

† These authors use similar techniques which, however, are based on background field methods, functional integration, etc which can be dispensed with.

Therefore

$$\int_0^\infty \mathrm{d}s[\kappa^2+p\,.\,(1+Xf^2X)\,.\,p-\tfrac{1}{2}\operatorname{Tr}(f^2X)]\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-M(s)}=1. \tag{3.12}$$

Setting

$$\begin{aligned} &1+Xf^2X=\partial X/\partial s,\\ &-\tfrac{1}{2}\operatorname{Tr}(f^2X)=\partial Y/\partial s, \end{aligned} \tag{3.13}$$

with boundary condition $\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-M(s)}|_0^\infty=-1$, we find as solution for the system (3.13):

$$\begin{aligned} &X(s)=f^{-1}\tan(fs),\\ &Y(s)=\tfrac{1}{2}\operatorname{Tr}\ln[\cos(fs)]. \end{aligned} \tag{3.14}$$

This yields the Green function in configuration space:

$$\Delta_+(x,x'|A')=\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-Y(s)}\int\frac{\mathrm{d}p}{(2\pi)^4}\exp[\mathrm{i}p(x-x')]\,\mathrm{e}^{-pX(s)p}. \tag{3.15}$$

Formula (2.10), however, requires only the diagonal part of $\Delta_+(x,x'|A')$ which we take from (3.15)

$$\Delta'_+(x,x)=\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-Y(s)}\int\frac{\mathrm{d}^np}{(2\pi)^n}\,\mathrm{e}^{-pXp}, \tag{3.16}$$

where the momentum space integral is taken to be n dimensional. Only at the end of the calculation do we take the limit $n=4$. The p integral occurring in (3.16) is readily evaluated:

$$\begin{aligned} \Delta'_+(x,x)&=\frac{1}{(2\pi)^n}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-Y(s)}\,\mathrm{i}(\sqrt{\pi})^n\frac{1}{(\det X)^{1/2}}\\ &=\frac{\mathrm{i}}{(4\pi)^{n/2}}\int_0^\infty\frac{\mathrm{d}s}{s^{n/2}}\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-Y(s)}\exp[-\tfrac{1}{2}\operatorname{Tr}\ln(s^{-1}X)]\\ &=\frac{\mathrm{i}}{(4\pi)^{n/2}}\int_0^\infty\frac{\mathrm{d}s}{s^{n/2}}\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-\tilde{y}(s)} \end{aligned} \tag{3.17}$$

with

$$\tilde{y}(s)=\tfrac{1}{2}\operatorname{Tr}\ln[(sf)^{-1}\sin(fs)]. \tag{3.18}$$

According to equation (3.3) we find

$$G_+(x,x|A)=m\frac{\mathrm{i}}{(4\pi)^{n/2}}\int_0^\infty\frac{\mathrm{d}s}{s^{n/2}}\,\mathrm{e}^{-s\kappa^2}\,\mathrm{e}^{-\tilde{y}(s)} \tag{3.19}$$

and

$$\begin{aligned} \operatorname{Tr}G_+(x,x|F)&=m\frac{\mathrm{i}}{(4\pi)^{n/2}}\int_0^\infty\frac{\mathrm{d}s}{s^{n/2}}\,\mathrm{e}^{-sm^2}\exp\{-\tfrac{1}{2}\operatorname{Tr}\ln[(esF)^{-1}\sin(eFs)]\}\operatorname{Tr}(\mathrm{e}^{\frac{1}{2}e\sigma Fs})\\ &=\mathrm{i}\frac{\partial\mathscr{L}^{(1)}}{\partial m}, \end{aligned} \tag{3.20}$$

which is stated in equation (2.10). A simple integration with respect to m produces Schwinger's old result for a constant field (Schwinger 1951):

$$\mathscr{L}^{(1)}[F] = -\frac{1}{2(4\pi)^{n/2}}\int_0^\infty \frac{ds}{s^{\frac{1}{2}n+1}}\,\mathrm{e}^{-sm^2}\,\mathrm{e}^{-\tilde{\gamma}(s)}\,\mathrm{tr}_4(\mathrm{e}^{\frac{1}{2}e\sigma Fs})$$
$$= \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\mathrm{e}^{-\mathrm{i}m^2s}\,\mathrm{e}^{-\gamma(s)}\,\mathrm{tr}_4(\mathrm{e}^{\frac{1}{2}\mathrm{i}e\sigma Fs}) \qquad \text{for } n=4,\ s\to \mathrm{i}s \tag{3.21}$$

and

$$\gamma(s) = \tfrac{1}{2}\,\mathrm{Tr}\,\ln[(eFs)^{-1}\sinh(eFs)]. \tag{3.22}$$

If there is only an external constant magnetic field present, e.g. $F_{12} = -F_{21} = H$, we find $\mathrm{Tr}\exp(\frac{1}{2}\mathrm{i}e\sigma Fs) = 4\cos(eHs)$ and

$$\exp(-\gamma(s)) = \det\left(\frac{eFs}{\sinh(eFs)}\right) = \frac{eHs}{\sin(eHs)}.$$

With the substitution $s \to -\mathrm{i}s$ in (3.21) and appropriate choice of contact terms, we find at last

$$\mathscr{L}^{(1)}_{\frac{1}{2}}[H] = -\frac{2}{(4\pi)^{n/2}}\int_0^\infty \frac{ds}{s^{\frac{1}{2}n+1}}\mathrm{e}^{-m^2s}[(eHs)\coth(eHs) - 1 - \tfrac{1}{3}(esH)^2] \tag{3.23}$$

$$= -\frac{2}{16\pi^2}\int_0^\infty \frac{ds}{s^3}\mathrm{e}^{-m^2s}[\cdots] \qquad \text{for } n=4. \tag{3.24}$$

If we make use of the formula (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty dz\, z^{\mu-1}\,\mathrm{e}^{-\alpha z}\coth z = \Gamma(\mu)(2^{1-\mu}\zeta(\mu,\alpha/2) - \alpha^{-\mu})$$

we can integrate (3.24) with the result

$$-V^{(1)}_{\frac{1}{2}}[H] \equiv \mathscr{L}^{(1)}_{\frac{1}{2}}[H] = -\frac{2}{64\pi^2}\left\{(2m_{\frac{1}{2}}^4 - 4m_{\frac{1}{2}}^2(eH) + \tfrac{4}{3}(eH)^2)\left[\ln\left(\frac{m_{\frac{1}{2}}^2}{eH}\right) + 1\right]\right.$$
$$\left. + 4m_{\frac{1}{2}}^2(eH) - 3m_{\frac{1}{2}}^4 + 2(4eH)^2\zeta'(-1;\, m_{\frac{1}{2}}^2/2eH)\right\} \tag{3.25}$$

where the subscript $\frac{1}{2}$ is indicative of spin-$\frac{1}{2}$ particles. One can now follow the same scheme as displayed so far to produce an equivalent result for scalar QED with external constant magnetic field:

$$\mathscr{L}^{(1)}_0[H] = \frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3}\mathrm{e}^{-m^2s}\left(\frac{eHs}{\sinh(eHs)} - 1 + \frac{1}{6}(esH)^2\right)$$
$$= \frac{1}{64\pi^2}\left\{[2m_0^4 - \tfrac{2}{3}(eH)^2]\left[1 + \ln\left(\frac{m_0^2}{eH}\right)\right] - 3m_0^4 + 2(4eH)^2\zeta'\left(-1;\,\frac{m_0^2 + eH}{2eH}\right)\right\}. \tag{3.26}$$

Here we employed the formula (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty dz\, z^{\mu-1}\,\mathrm{e}^{-\alpha z}\frac{1}{\sinh z} = 2^{1-\mu}\Gamma(\mu)\zeta(\mu;\tfrac{1}{2}(\alpha+1)).$$

Adding up the two contributions then leads to

$$\mathscr{L}^{(1)}_{0+\frac{1}{2}}[H]=\frac{1}{64\pi^2}\bigg[[2m_0^4-\tfrac{2}{3}(eH)^2]\ln\Big(\frac{m_0^2}{eH}\Big)+[-4m_{\frac{1}{2}}^4+8m_{\frac{1}{2}}^2(eH)-\tfrac{8}{3}(eH)^2]\ln\Big(\frac{m_{\frac{1}{2}}^2}{eH}\Big)$$
$$-m_0^4-\tfrac{2}{3}(eH)^2+2(4eH)^2\zeta'\Big(-1;\frac{m_0^2+eH}{2eH}\Big)$$
$$+2m_{\frac{1}{2}}^4-\tfrac{8}{3}(eH)^2-4(4eH)2\zeta'\Big(-1;\frac{m_{\frac{1}{2}}^2}{2eH}\Big)\bigg]. \qquad (3.27)$$

The other interesting case, in which only a pure electric field is applied, yields an effective Lagrangian

$$\mathscr{L}^{(1)}_{\frac{1}{2}}[E]=\mathscr{L}^{(1)}_{\frac{1}{2}}[H\to \mathrm{i}^{-1}E \text{ in } (3.25)].$$

In contrast with (3.25) this Lagrangian has an imaginary part which leads to a non-vanishing probability of the vacuum remaining unchanged: $|\langle 0_+|0_-\rangle|^2=\exp(-2\,\mathrm{Im}\,W^{(1)})$. The result for the production probability of an electron-positron pair in an external electric field is exactly that given by Schwinger (1951).

If the external electromagnetic field is taken to be a laser field, it is also possible to write down the exact solution for the basic Green function equation (3.1) (Schwinger 1951, Dittrich 1972, Mitter 1975):

$$G_+(x', x''|A^{\mathrm{L}})=(m-\gamma\pi)\,\mathrm{e}^{\mathrm{i}e\phi}\frac{1}{(4\pi)^2}\int_0^\infty\frac{\mathrm{d}s}{s^2}\exp\Big[\mathrm{i}\Big(\frac{(x'-x'')^2}{4s}-(m^2+e^2\delta m^2)s\Big)\Big]$$
$$\times\Big(1+\mathrm{i}s\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}\mathrm{d}\xi\,\tfrac{1}{2}ea\sigma f^{\lambda}F_{\lambda}(\xi)\Big)$$
$$=\mathrm{e}^{\mathrm{i}e\phi(x',x'')}\Big\{m-\gamma\Big[\frac{1}{\mathrm{i}}\partial'_\mu+f_{\mu\nu}(x'-x'')^\nu\frac{1}{\xi'-\xi''}\Big(eA(\xi')-\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}\mathrm{d}\xi eA(\xi)\Big)\Big]\Big\}$$
$$\times\Delta_+(x'-x''|A') \qquad (3.28)$$

where

$$\delta m^2(\xi', \xi''|A_\lambda)=\frac{a^2}{\xi'-\xi''}\int_{\xi''}^{\xi'}\mathrm{d}\xi\,A_\lambda(\xi)A^\lambda(\xi)-\frac{a^2}{(\xi'-\xi'')^2}\int_{\xi''}^{\xi'}\mathrm{d}\xi\,A_\lambda(\xi)\int_{\xi''}^{\xi'}\mathrm{d}\eta\,A^\lambda(\eta)\geqslant 0.$$

The remainder of the definitions can be looked up in Dittrich (1972) and Mitter (1975). Again we want to extract the diagonal part $G_+(x, x|A^{\mathrm{L}})$ and therefore need to know the behaviour of $\delta m^2(\xi', \xi'')$ for $x'\simeq x''$†. This can be done by changing variables

$$\xi=\frac{1+\lambda}{2}\xi'+\frac{1-\lambda}{2}\xi''=X+\lambda z, \qquad \mathrm{d}\xi=z\,\mathrm{d}\lambda,$$

where $X=\frac{1}{2}(\xi'+\xi'')$, $z=\frac{1}{2}(\xi'-\xi'')$, and expanding

$$A(X+\lambda z)\simeq A(X)+\lambda zA'(X)+\tfrac{1}{2}(\lambda z)^2A''(X)$$
$$=A(X)+\lambda zF(X)+\tfrac{1}{2}(\lambda z)^2A''(X).$$

† I am indebted to Wu-yang Tsai of the University of California at Los Angeles on this point.

Then we obtain the following short-distance behaviour of δm^2:

$$\delta m^2(\xi', \xi'') = a^2\left[\int_{-1}^{+1} \tfrac{1}{2}\mathrm{d}\lambda\, A^2(X+\lambda z) - \left(\int_{-1}^{+1} \tfrac{1}{2}\mathrm{d}\lambda\, A(X+\lambda z)\right)^2\right]$$

$$\cong a^2\left[\int_{-1}^{+1} \tfrac{1}{2}\mathrm{d}\lambda\, [A^2(X) + \lambda^2 z^2(A'(X)^2 + A(X)A''(X))]\right.$$

$$\left. - \left(\int_{-1}^{+1} \tfrac{1}{2}\mathrm{d}\lambda\, (A(X) + \tfrac{1}{2}\lambda^2 z^2 A''(X))\right)\right]$$

$$= A^2(X) + \tfrac{1}{3}z^2(A'^2 + AA'') - (A^2 + \tfrac{1}{3}z^2 A''A)$$

$$= \tfrac{1}{12}a^2(\xi' - \xi'')^2 F^2[\tfrac{1}{2}(\xi' + \xi'')] = -\tfrac{1}{12}a^2(x' - x'')^\mu F_\mu{}^\lambda F_{\lambda\nu}(x' - x'')^\nu.$$

Therefore, we find for $x' \simeq x''$,

$$\Delta'_+(x' - x'') \underset{x' \simeq x''}{=} \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}m^2 s}\exp\{\tfrac{1}{4}\mathrm{i}s^{-1}(x' - x'')^\mu[g_{\mu\nu} + \tfrac{1}{3}(eas)^2 F_\mu{}^\lambda F_{\lambda\nu}](x' - x'')^\nu\}$$

$$\times \underbrace{\left(1 + \mathrm{i}s\frac{1}{\xi' - \xi''}\int_{\xi''}^{\xi'} \mathrm{d}\xi\, \tfrac{1}{2}ea\sigma f^\lambda F_\lambda(\xi)\right)}_{1 + \mathrm{i}s\int_{-1}^{+1} \frac{1}{2}\mathrm{d}\lambda'\, \frac{1}{2}ea\sigma f^\lambda F_\lambda[X + \lambda' z].}$$

Now we can take the limit $x' \to x''$ in equation (3.28) and obtain the result:

$$G_+(x, x|A^{\mathrm{L}}) = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}m^2 s}\left(m - \tfrac{1}{4}\gamma . k\frac{\partial}{\partial\xi}\right)(1 + \tfrac{1}{2}\mathrm{i}ea\sigma f^\lambda F_\lambda(\xi)s). \tag{3.29}$$

The solution of the differential equation $\mathrm{i}\partial\mathscr{L}^{(1)}/\partial m = \mathrm{tr}\, G_+(x, x)$ is now given by

$$\mathscr{L}^{(1)} = \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-\mathrm{i}m^2 s} + \mathrm{CT}$$

where the contact term (CT) has to be chosen so as to produce a vanishing result for $\mathscr{L}^{(1)}$.

The one-loop effective potential is therefore strictly zero, contrary to the constant magnetic field. This result also holds when the particle that propagates through the external laser field is taken to be a scalar. Here the Green function, for $x' \to x''$, is simply

$$\Delta_+(x, x) = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\mathrm{i}m_0^2 s} + \mathrm{CT}$$

and the differential equation to be solved is given by

$$-\mathrm{i}\partial\mathscr{L}^{(1)}/\partial m_0^2 = \Delta_+(x, x)$$

which yields

$$\mathscr{L}^{(1)} = -\frac{1}{16\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-\mathrm{i}m_0^2 s} + \mathrm{CT}.$$

Hence the laser field is not a suitable environment to generate a non-vanishing effective potential (in the one-loop approximation) (Salam and Strathdee 1975, Kibble *et al* 1975).

There is, however, the possibility of creating an effective potential if we take a pure scalar field theory instead of a vector theory. Here we present only the most important steps. Let the scalar plane wave field be $A = A(\xi)$, with $\xi = k \, . \, x$, $k^2 = 0$, and the Lagrangian be taken as

$$\mathscr{L} = -\tfrac{1}{2}[(\partial_\mu \phi)^2 + (m + gA)^2 \phi^2].$$

The basic Green function equation is now given by

$$(-\partial^2 + m^2 + gJ)\Delta_+[J] = \delta,$$

where $J = 2mA + gA^2$. The solution of this equation can be found with the *ansatz*

$$\Delta_+(x, p|J) = \int \mathrm{d}y \, \mathrm{e}^{\mathrm{i}py} \Delta_+(x, y|J) = \mathrm{e}^{\mathrm{i}px} g_+(x, p|J)$$

with

$$[m^2 + (p + \mathrm{i}^{-1}\partial)^2 + gJ]g_+ = (p^2 + m^2 + 2p\mathrm{i}^{-1}\partial + gJ)g_+ = 1 \tag{3.30}$$

having anticipated $(\mathrm{i}^{-1}\partial)^2 g_+ = 0$, since $k^2 = 0$. Equation (3.30) can be satisfied by

$$g_+(x, p|J) = \mathrm{i}\int_0^\infty \mathrm{d}\alpha \, \exp[-\mathrm{i}\alpha(p^2 + m^2 - \mathrm{i}\epsilon)] \exp\left(-\mathrm{i}g \int_0^\alpha \mathrm{d}\alpha' J(x - 2p\alpha')\right),$$

or, if we perform the Fourier transform on the outgoing variable,

$$g_+(q, y|J) = \mathrm{i}\int_0^\infty \mathrm{d}\alpha \, \exp[-\mathrm{i}\alpha(q^2 + m^2 - \mathrm{i}\epsilon)] \exp\left(-\mathrm{i}g \int_0^\alpha \mathrm{d}\alpha' J(y + 2q\alpha')\right).$$

We then obtain

$$\Delta_+(x', x''|J) = \int \frac{\mathrm{d}q}{(2\pi)^4} \mathrm{e}^{\mathrm{i}q(x'-x'')} \mathrm{i}\int_0^\infty \mathrm{d}\alpha \, \exp[-\mathrm{i}\alpha(q^2 + m^2)] \exp\left(-\mathrm{i}g \int_0^\alpha \mathrm{d}\alpha' J(x'' + 2q\alpha')\right).$$

By various changes of variables $\Delta_+(x', x''|J)$ can be converted into (Dittrich 1972, Mitter 1975)

$$\Delta_+(x', x''|A) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} \exp\left[\mathrm{i}\left(\frac{(x' - x'')^2}{4\alpha} - (m^2 + \delta m^2)\alpha\right)\right] \tag{3.31}$$

where

$$\delta m^2(\xi', \xi''|A) = \frac{2mg}{\xi' - \xi''} \int_{\xi''}^{\xi'} \mathrm{d}\xi A(\xi) + \frac{g^2}{\xi' - \xi''} \int_{\xi''}^{\xi'} \mathrm{d}\xi A^2(\xi). \tag{3.32}$$

For a simple choice of $A(\xi)$, e.g. $A(\xi) = a\xi$, $\xi = k \, . \, x$, we get

$$\delta m^2(\xi', \xi'') = mga(\xi' + \xi'') + \tfrac{1}{3}g^2 a^2[(\xi' + \xi'')^2 - \xi'\xi''].$$

The behaviour of $\delta m^2(\xi', \xi'')$, for $\xi' \simeq \xi''$, as is important for vacuum polarization phenomena, is given by

$$\delta m^2(\xi) = 2mgA(\xi) + g^2 A(\xi).$$

The diagonal element of Δ_+ then yields

$$\Delta_+(x, x|A) = \frac{1}{16\pi^2} \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} \mathrm{e}^{-\mathrm{i}\kappa^2\alpha} + \mathrm{CT},$$

where $\kappa^2 = m^2 + \delta m^2(\xi)$.

On employing $i\partial\mathscr{L}^{(1)}/\partial\kappa^2 = -\frac{1}{2}\Delta_+(x, x|J)$ we find at last

$$\mathscr{L}^{(1)}[A] = -\frac{1}{32\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^3} e^{-i\alpha m^2}[e^{-i\alpha\delta m^2(\xi)} - 1 + i\alpha\delta m^2(\xi) + \tfrac{1}{2}\alpha^2(\delta m^2(\xi))^2]$$

which leads to the following effective potential:

$$-\mathscr{L}^{(1)} \equiv V^{(1)}[A] = \frac{1}{64\pi^2}\bigg[(m^2 + 2mgA(\xi) + g^2A^2(\xi))^2 \ln\left(1 + \frac{2mgA(\xi) + g^2A^2(\xi)}{m^2}\right) - 2m^3gA(\xi) - g^2m^2A(\xi) - \tfrac{3}{2}(2mgA(\xi) + g^2A^2(\xi))^2\bigg].$$

4. Conclusion

Our main goal in this paper was to set up a Green function formalism for computing effective Lagrangians in the one-loop approximation. This has been achieved for scalar and spinor QED with constant magnetic (electric) field. Finally, some of the conceptual difficulties associated with the generation of masses in the one-loop approximation (instead of spontaneous symmetry breaking) have been clarified for the laser case. Laser field theory is not the right arena for setting up a mass generating mechanism (in the one-loop approximation). This fact is in contrast with the result of a pure neutral scalar field theory. Here one finds an effective potential (mass shift) that a scalar particle would experience when propagating through an external c-number field.

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